Distribution of linear statistics in random matrix models (metallic conductance fluctuations)

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Distribution of linear statistics in random matrix models

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Abstract. In this paper the distribution of linear statistics in random matrix models is evaluated in the large-N approximation. The distribution function is found to be a Gaussian, but with a mean and mode, which depend on the single-particle potential, that do not coincide, albeit with a vanishingly small discrepancy in the limit of a perfect conductor.

Since the discovery of universal conductance fluctuations in mesoscopic conductors it has become clear that the conductance and other linear statistics are non-self-averaging quantities, e.g. the mean and the mode of the distribution function are different. Therefore, physical quantities, such as the conductance and magneto-conductance, should be described in terms of their distribution functions rather then their moments and so in this letter we shall concentrate upon these distribution functions and derive them, using random matrix theory, for the metallic regime.

Random matrix theory, originally conceived to provide a theoretical description of the energy levels of heavy nuclei [1, 2], but recently applied to other diverse areas of physics such as quantum chaos [3] and string theory [4], provides a framework for such a description, based upon the models of random transfer matrices of Imry [5, 6]. In these models a disordered conductor of length L and cross-sectional area L^{d-1} with $N = (k_F L)^{d-1}$—k_F the Fermi momentum—propagating channels is parametrized by a $2N \times 2N$ transfer matrix $\mathbf{T}$. Physical quantities are then given by functions of the eigenvalues x_n (> 0), $n = 1, \ldots, N$, of the matrix $\mathbf{X} = \frac{1}{4}(\mathbf{T}^\dagger\mathbf{T} + (\mathbf{T}^\dagger\mathbf{T})^{-1} - 2)$, which, as with all random matrices, may be orthogonal, unitary or symplectic. We shall be interested in 'linear statistics', where a typical physical quantity f is given by $f = \sum_n f(x_n)$, e.g. the two-probe conductance g, related to the eigenvalues by $g = \sum_n (1 + x_n)^{-1}$.

According to the maximum entropy *ansatz* [6] the joint probability distribution of x_n is

$$P[x_1, \ldots, x_N] \propto \mathrm{e}^{-E[V]} \tag{1}$$

where

$$E[V] = -\beta \sum_{1 \leqslant m < n \leqslant N} \ln|x_m - x_n| + \sum_{1 \leqslant n \leqslant N} V(x_n) \tag{2}$$

$\beta = 2, 1, 4$ describes respectively the unitary, orthogonal and symplectic ensemble and $V(x)$ is the confining potential.

A critical quantitative test of the validity of the random transfer matrix approach in the weak-disorder regime was recently carried out [7] by comparing an exact analytic calculation, within the random matrix theory, of the two-level cluster function for a given finite-size system, with an independent numerical evaluation of the cluster function using the tight-binding Anderson model. The excellent overall agreement demonstrates the general validity of the random matrix model. This confirms the hypothesis [5] that the

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only correlation in the joint probability distribution is a logarithmic interaction between eigenvalues which is universal and is completely determined from symmetry considerations alone. The 'single-particle' potential $V(x)$, which may be thought of as a Lagrange multiplier function from some additional constraint, e.g. the given mean value of the conductance, depends on the system parameters. Although this single-particle potential is not known from any microscopic Hamiltonian, attempts have been made [6, 8] to consider simple potentials suggested in the metallic regime. For example it is now well understood that a linear confining potential ($V(x) = tx$) with a sufficiently large slope (t is the Drude conductance) describes the metallic regime very well. The correlation of the eigenvalues due to this strongly confining potential results in a universal variance of the conductance [6, 10]. This was first discovered from a diagrammatic perturbation calculation in the metallic region and infinite-volume limit [9] and the above results are in good qualitative agreement with it. In the large-N limit, we may treat as a fluid the collection of eigenvalues [2]. In this approximation, the universal variance was found and an analogue of the Dyson–Mehta theorem [2] for linear statistics was derived recently [11].

In view of the accuracy of the fluid approximation in the metallic region (the eigenvalue density scales with N) [2, 12, 13, 14, 15], we shall apply it to the problem of determining the distribution function for linear statistics.

Consider the distribution function of a linear statistics defined as

$$P(f) = \left\langle \delta\left(f - \sum_{n=1}^{N} f(x_n)\right)\right\rangle \tag{3}$$

where $\langle \ldots \rangle$ denotes average with respect to $\mathrm{e}^{-E[V]}$. To determine $P(f)$, we may apply a parametric representation of the δ-function $\delta(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \mathrm{d}k \exp[-\mathrm{i}kx]$ and write

$$P(f) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \bar{P}(k)\mathrm{e}^{\mathrm{i}kf} \tag{4}$$

where the moment generating function is

$$\bar{P}(k) := \int \mathrm{d}\mu(x)\, \mathrm{e}^{-E[V+V_{\mathrm{ex}}]} \Big/ \int \mathrm{d}\mu(x)\mathrm{e}^{-E[V]} \tag{5}$$

and where $V_{\mathrm{ex}}(x) = \mathrm{i}kf(x)$ is the 'external potential' and $\int \mathrm{d}\mu(x) := \int_0^\infty \ldots \int_0^\infty \prod_{n=1}^{N} \mathrm{d}x_n$. Thus, $-\ln \bar{P}(k)$ is the change of the free energy due to the perturbation of the 'external potential'.

In the large-N limit, the free energy, $E[U]$, reads

$$\bar{E}[U] = -\frac{\beta}{2}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \sigma(x)\ln|x-y|\sigma(y) + \int_0^\infty \mathrm{d}x\, [V(x)+C]\sigma(x) \tag{6}$$

with $\int_0^\infty \mathrm{d}x\, \sigma(x) = N$ and C the chemical potential for the particle number constraint. The equilibrium density, $\sigma(x)$, therefore satisfies an integral equation of Dyson [16],

$$\beta \int_0^\infty \mathrm{d}t\ \sigma(t)\ln|x-t| = V(x) + C. \tag{7}$$

The solution of an equivalent equation

$$-\mathrm{P}\int_0^\infty \mathrm{d}t\ \frac{\sigma(t)}{t-x} = \frac{1}{\beta}V'(x) \tag{8}$$

found by applying $\mathrm{d}/\mathrm{d}x$ on (7) reads [17]

$$\sigma(x) = \frac{1}{\pi^2\beta}\frac{1}{\sqrt{x}}\mathrm{P}\int_0^\infty \frac{\mathrm{d}t}{t-x}\sqrt{t}\ U'(t) \tag{9}$$

and can be recast in the following form [18]:

$$\sigma(x;[U]) = \frac{1}{\pi^2\beta}\int_0^\infty \mathrm{d}y\ G(x,y)V(y) \tag{10}$$

where $G(x,y) := \partial_x\partial_y \ln\left[|\sqrt{x}-\sqrt{y}|/(\sqrt{x}+\sqrt{y})\right]$; note that this was derived in [11]. With these it is easily seen that $\bar{P}(k) = (1/\mathcal{N})\mathrm{e}^{-\delta\bar{E}}$, where $\mathcal{N}$ is the normalization constant to be determined by setting $\int \mathrm{d}\mu(f)\ P(f) = 1$ and $\delta\bar{E} := \bar{E}[V+V_{\mathrm{ex}}] - \bar{E}[V]$. Thus, because σ is linearly related to U, $\sigma(x;[V+V_{\mathrm{ex}}]) = \sigma_0 + \sigma_1$, where σ_0 and σ_1 depend, respectively, only on V and V_{ex} and therefore

$$\delta\bar{E} = -\frac{k^2}{2\pi^2\beta}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\ f(\sqrt{x})G(x,y)f(\sqrt{y}) + \frac{\mathrm{i}k}{\pi^2\beta}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\ f(\sqrt{x})G(x,y)V(\sqrt{y}) \tag{11}$$

where we have used (9) and (10) to simplify the expression and assumed, without loss of generality, that f and V are functions of $\sqrt{x}$, as will become clear later.

With the redefinition $\sqrt{x} = t_1$ and $\sqrt{y} = t_2$, followed by integration by parts in t_1, t_2, we find by a Mellin transform that $\delta\bar{E} = -Ak^2/2 + \mathrm{i}Bk$, where

$$A := \frac{2}{\pi^2\beta}\int_{-\infty}^\infty \mathrm{d}\mu\ \mu\tanh(\pi\mu)|\overline{f}(2\mathrm{i}\mu)|^2 \tag{12}$$

$$B := \frac{2}{\pi^2\beta}\int_{-\infty}^\infty \mathrm{d}\mu\ \mu\tanh(\pi\mu)\overline{f}(-2\mathrm{i}\mu)\overline{V}(2\mathrm{i}\mu) \tag{13}$$

and $\overline{F}(\mu) := \int_0^\infty \mathrm{d}x\ x^{\mu-1}F(x)$, i.e. $\overline{F}$ is the Mellin transform of F. Equation (12) has been recently found in [11] and also through a calculation using orthogonal polynomial techniques in the large-N, followed by the large-t, limit, based on the Bessel kernel [10, 19]. Thus, upon performing the inverse Fourier transform on $\overline{P}$ we obtain

$$P(f) = \frac{1}{\mathcal{N}}\exp\left[-\frac{(f-B)^2}{2A}\right] \tag{14}$$

which is a universal distribution function with its mean value dependent on the type of linear statistics and the 'internal potential' V. This distribution is valid only in the metallic region where there is good conduction. Note that, using random matrix theory, it has been previously argued by Politzer [20] that a Gaussian distribution should follow, but his 'formal' method of derivation was unable to give an explicit result for the variance, unlike the present letter.

Given the distribution function for the linear statistic f it is straightforward to determine its mean and variance. If f is distributed along the whole of the real axis, then from (14) the mean of f is $\overline{f} = B$ and the variance squared is $\overline{(\Delta f)^2} = A$, where $\Delta f = f - \overline{f}$. The variation is therefore universal, depending only upon the linear statistic and not upon the particular system, i.e. the internal potential. However, linear statistics typically only take positive values, in which case the mean of the distribution will no longer equal the mode, i.e. $\overline{f} \neq B$ and the variance will depend upon the internal potential. However, as we will now show, the discrepency between the mean and the mode may well be extremely small and, in any case, this will certainly be so when in the metallic regime.

As an example of the above, let us consider the conductance (g), where $f(x) = 1/(1+x)$. It follows that $A = 1/8\beta$, but to determine B we require a particular internal potential. If we take $V(x) = x^a$, $0 < a < \frac{1}{2}$, then $B = a/(\beta\cos(\pi a))$, which is positive. It follows that the mean of g and g^2 can be easily expressed in terms of incomplete gamma

functions, but simple expressions follow for the variance squared of g in two limits of a. As $a \to 0$, $B \to 0$ too, so the distribution tends to a Gaussian centred at the origin and therefore $\overline{(\Delta g)^2} \to (1 - 2/\pi)A$. It should be noted that, while in this limit the difference between the mean and the mode take its maximum value, the mean and variance of the conductance become comparable in magnitude and we are therefore no longer in the metallic regime. We therefore expect $P(g)$ to take on a different structure as we are then approaching the insulating regime. Near the insulating regime, $V(x) \sim [\ln(x)]^2$, $x \gg 1$ and the potential we have chosen fails to describe this situation; see [7] and [15]. In the opposite limit of $a \to \frac{1}{2}$, B diverges and the expectation values with respect to this distribution should then become insensitive to the fact that only positive values of g are taken since the Gaussian will only have appreciable weight within the variance, about the mode. Therefore, in this limit, which is the metallic regime, $\overline{(\Delta g)^2} \to A = 1/8\beta$, which is the same result as found by other means. Also, the mode of the distribution diverges, i.e. the system tends towards being a perfect conductor and the difference between the mean and mode vanish in the limit. However, it was observed in [11] that there is deviation from the logarithmic repulsions between the eigenvalues.

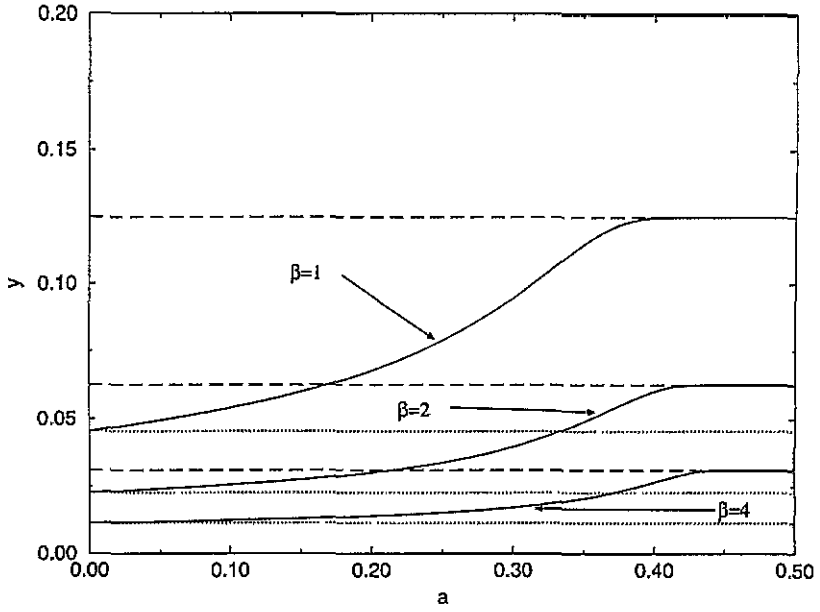

Figure 1. A plot of the square of the variance of the conductance, $\overline{(\Delta g)^2}$, for $\beta = 1, 2$ and 4, along with its limiting values, for the single-particle potential $V(x) = x^a$. The solid line is $y = \overline{(\Delta g)^2}$ as a function of a, the dotted line is $y = (1 - 2/\pi)/8\beta$ and the dashed line is $y = 1/8\beta$.

The variance squared, along with its limiting values, is displayed in figure 1 for all three values of β. In figure 2 $P(g)$ itself is displayed, for $a = 0.3, 0.4, 0.45$ and 0.47, all with $\beta = 2$. Note that for $a = 0.3$ we are no longer in the metallic regime, but we include the curve for reference as it is very clear that here the mean and variance are comparable, as noted above.

In the same way we can determine the distribution of the number of eigenvalues in the interval $(0, s)$, $n(s)$, in the large-N limit for the internal potential used above. Since, $n(s) = \int_0^s \mathrm{d}x\ \sigma(x)$, as a linear statistics we have $f(x) = \Theta(s - x)$, and so $n(s)$ is

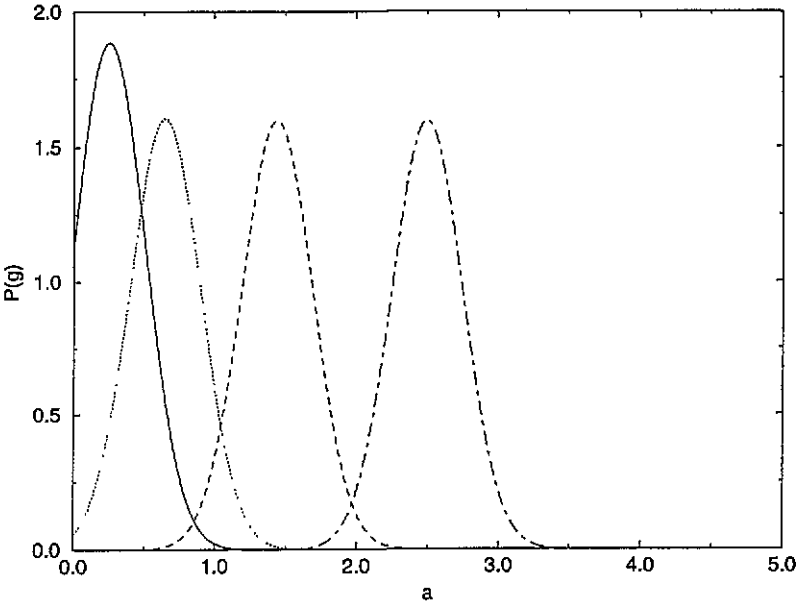

Figure 2. A plot of the distribution function of the conductance for the internal potential $V(x) = x^a$. The solid line is $a = 0.3$, the dotted line is $a = 0.4$, the dashed line is $a = 0.45$ and the dot-dashed line is $a = 0.47$, all with $\beta = 2$.

distributed according to the normal distribution of (14), with $A = (1/2\pi^2\beta)\ln(s/\epsilon))$ and $B = (1/\pi\beta)\tan(\pi a/4)\, s^{a/4}$, where ϵ is a regulator that cuts off the short-distance divergence introduced in the fluid approximation.

In conclusion, in the large-N limit of random matrix theory, using the Coulomb fluid approximation, we have derived the distribution function of an arbitrary linear statistic, finding it to be Gaussian, as previously predicted [20], but now being able to obtain expressions for the variance and mean. Taking care to note the conditions on the potential introduced in obtaining (10), we have explicitly evaluated the distribution of the conductance for the potential $V(x) = x^a$, $0 < a < \frac{1}{2}$. Using this potential, we have also evaluated the distribution for the $n(s)$, the number of eigenvalues in the range $(0, s)$.

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[15] The fluid approximation is valid for sufficiently strong confining potentials, where the eigenvalue density scales with N, the channel number. This arises in the bulk of the densities of random matrix ensembles associated with classical orthogonal polynomials; see
Nagao T and Wadati M 1991 *J. Phys. Soc. Japan* **60** 3298
and is expected to break down when the confining potential is marginal, e.g. $V(x) \sim (\ln x)^2$ for large values of x. In this situation it can shown that there is a macroscopic reduction $[\emptyset(N)]$ in the density. This occurs in the unitary q−Laguerre ensemble; see
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The original equation has an addtional non-linear term $(\beta - 2)\ln\sigma$ in the left-hand side, which vanishes for unitary ensembles. For other ensembles, the non-linear term is a small perturbation of relative order $(\ln N)/N$, and can be discarded as $N \to \infty$; however, see [15].
[17] Equation (7) appears in certain electrostatic problems where $\sigma(x)$ has the physical meaning of the induced charge density on a grounded thin conducting half-plane, introduced into a given plane-parallel external electric field; see
Lebedev N N, Skalskaya I P and Uflyand Y S 1965 *Worked Problems In Applied Mathematics* (New York: Dover) problem 556, pp 266, 267
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Akhiezer N I and Glazman I M *Theory of Linear Operators in Hilbert Space* vol 1 (New York: Ungar) pp 114, 115
[18] This is found by an integration by parts. For the boundary contribution to be discarded, we demand that $U(x)\sqrt{x}$ vanishes at the origin and $U(x)/\sqrt{x}$ vanishes at infinity. This, to a large extent, determines the choice of the single-particle potential, $V(x)$, which includes the potential discovered in [7].
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